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AN APPROXIMATION OF INTEGRABLE FUNCTIONS BY STEP FUNCTIONS WITH--ETC(U)
JAN 79 M G CRANDALL, A PAZY DAAG29-75-C-0024

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MRC Technical Summary Report #1911

AN APPROXIMATION OF INTEGRABLE FUNCTIONS BY STEP FUNCTIONS WITH AN APPLICATION

M. G. Crandall and A. Pazy

See 1473 in back.

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January 1979

(Received July 6, 1978)

Approved for public release
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U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

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AN APPROXIMATION OF INTEGRABLE FUNCTIONS BY STEP FUNCTIONS WITH AN APPLICATION

M. G. Crandall[1)] and A. Pazy[1),2)]

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ABSTRACT

Let $f \in L^1(0,\infty)$, $\delta > 0$ and $(G_\delta f)(t) = \delta^{-1} \int_t^\infty e^{(t-s)/\delta} f(s)ds$. Given a partition $P = \{0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots\}$ of $[0,\infty)$ where $t_i \to \infty$, we approximate f by the step function $A_P f$ defined by

$$A_P f(t) = (G_{\delta_i} G_{\delta_{i-1}} \cdots G_{\delta_1} f)(0) \quad \text{for} \quad t_{i-1} \le t < t_i ,$$

where $\delta_i = t_i - t_{i-1}$. The main results concern several properties of this process, with the most important one being that $A_P f \to f$ in $L^1(0,\infty)$ as $\mu(P) = \sup_i \delta_i \to 0$. An application to difference approximations of evolution problems is sketched.

AMS (MOS) Subject Classification: 41A30
Key Words: Step functions, Approximation theory, Accretive operators
Work Unit Number 6 (Spline Functions and Approximation Theory)

Sponsored by:
1) The United States Army under Contract No. DAAG29-75-C-0024; and
2) The National Science Foundation under Grant No. MCS78 01245.

SIGNIFICANCE AND EXPLANATION

A method of approximating functions which are integrable on $(0,\infty)$ by piecewise constant functions is presented and studied
L infinity
in this paper. The method used and the properties established for it allow one to reduce the study of the convergence of a difference method of theoretical interest for nonlinear time-dependent problems with forcing terms to the simpler study of related problems without forcing terms.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

AN APPROXIMATION OF INTEGRABLE FUNCTIONS BY STEP

FUNCTIONS WITH AN APPLICATION

M. G. Crandall[1)] and A. Pazy[1),2)]

This note is concerned with an interesting method of approximating an integrable function $f : (0,\infty) \to \mathbb{R}$ by step functions. The approximation process involves the integral transformation $G_\delta : L^1(0,\infty) \to L^1(0,\infty)$ defined for $\delta > 0$ by

(1) $$(G_\delta f)(t) = \frac{1}{\delta}\int_t^\infty e^{(t-s)/\delta} f(s)ds .$$

Equivalently, $g = G_\delta f$ is the unique function $g \in L^1(0,\infty)$ which satisfies $g - \delta g' = f$.

Let

$$P = \{0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots\}$$

be a partition of $[0,\infty)$ with $\lim_{i\to\infty} t_i = \infty$. The step sizes of the partition are denoted by δ_i; $\delta_i = t_i - t_{i-1}$. Each partition P determines a piecewise constant approximation $A_P f$ of f defined by

(2) $$A_P f(t) = (G_{\delta_i} G_{\delta_{i-1}} \cdots G_{\delta_1} f)(0) \quad \text{for } t_{i-1} \le t < t_i, \qquad i = 1,2,\ldots .$$

The mesh of the partition is denoted by $\mu(P)$; $\mu(P) = \sup_{1\le i<\infty} \delta_i$. The main results are summarized in the following theorem.

Theorem: Let P be as above, $f \in L^1(0,\infty)$ and A_P be defined by (2). Then

(3) $$A_P f \in L^1(0,\infty) ,$$

(4) $$\int_0^\infty |A_P f(s)|ds \le \int_0^\infty |f(s)|ds ,$$

(5) $$\int_0^\infty A_P f(s)ds = \int_0^\infty f(s)ds ,$$

and

(6) $$\lim_{\mu(P)\to 0} \int_0^\infty |A_P f(s) - f(s)|ds = 0 .$$

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The definition of the transformation $f \to A_P f$ as well as the questions resolved by the theorem arose naturally from considering difference approximations of certain nonlinear evolution problems. While this motivation is not relevant for the statement or the proof (given in Section 1) of the theorem, we do explain it briefly in Section 2.

We are indebted to Carl de Boor for his advice on this problem.

Section 1. (Proof of the Theorem).

Let $h,k \in L^1(-\infty,0)$ and $f \in L^1(0,\infty)$. We define $k \circ f \in L^1(0,\infty)$ and $h * k \in L^1(-\infty,0)$ according to

$$(k \circ f)(t) = \int_t^\infty k(t-s)f(s)ds \tag{1.1}$$

and

$$h * k(r) = \int_r^0 h(r-s)k(s)ds \ . \tag{1.2}$$

The convolution operator "$*$" is commutative and associative, while

$$h \circ (k \circ f) = (h * k) \circ f \ . \tag{1.3}$$

For $\delta > 0$ we set

$$k_\delta(r) = \delta^{-1}\exp(r/\delta) \ . \tag{1.4}$$

The transformation G_δ in (1) is

$$G_\delta f = k_\delta \circ f \ . \tag{1.5}$$

Let $P = \{0 = t_0 < t_1 < \cdots < t_i < t_{i-1} < \cdots\}$ and $\delta_i = t_i - t_{i-1}$ be as in the introduction and A_P be given by (2). For simplicity of notation we will set

$$k_i = k_{\delta_i} \quad \text{and} \quad K_i = k_i * K_{i-1} = k_i * k_{i-1} * \cdots * k_1 \quad \text{for} \quad i = 1,2,\ldots \ . \tag{1.6}$$

Since $k_i \geq 0$, A_P clearly satisfies

$$|A_P f| \leq A_P|f| \ . \tag{1.7}$$

Moreover by (2), (1.5), (1.6) and (1.3)

$$\begin{aligned}\int_0^{t_i} A_P|f|(s)ds &= \sum_{\ell=1}^{i} \delta_\ell(k_\ell \circ (k_{\ell-1} \circ (\cdots \circ (k_2 \circ (k_1 \circ |f|))\ldots)))(0) \\ &= \sum_{\ell=1}^{i} \delta_\ell(K_\ell \circ |f|)(0) = \int_0^\infty \sum_{\ell=1}^{i} \delta_\ell K_\ell(-s)|f(s)|ds \ .\end{aligned} \tag{1.8}$$

Since each of the summands $\delta_\ell K_\ell(-s)$ in the last integrand is nonnegative, we can establish (3) and (4) of the theorem by showing that

$$\sum_{\ell=1}^{\infty} \delta_\ell K_\ell(r) \le 1 \quad \text{for} \quad -\infty < r < 0 \tag{1.9}$$

while (5) requires

$$\sum_{\ell=1}^{\infty} \delta_\ell K_\ell(r) = 1 \quad \text{a.e. on} \quad -\infty < r < 0 \ . \tag{1.10}$$

The following lemma implies (1.9) since $K_j * 1 \ge 0$.

<u>Lemma 1</u>: For each $j = 1,2,\ldots$

$$\sum_{\ell=1}^{j} \delta_\ell K_\ell + K_j * 1 \equiv 1 \ . \tag{1.11}$$

<u>Proof of Lemma 1</u>: We proceed by induction. If $j = 1$ the claim is that $\delta_1 k_1 + k_1 * 1 \equiv 1$. Indeed, for any δ,

$$\delta k_\delta + k_\delta * 1 = e^{r/\delta} + \frac{1}{\delta}\int_r^0 e^{(r-s)/\delta} ds \equiv 1 \ . \tag{1.12}$$

We now assume the claim is true for $j = i$ and verify it for $j = i + 1$. By (1.12) we have

$$K_{i+1}*1 = k_{i+1}*K_i*1 = k_{i+1}*1*K_i = (1-\delta_{i+1}k_{i+1})*K_i = K_i*1 - \delta_{i+1}K_{i+1} = 1 - \sum_{\ell=1}^{i+1} \delta_\ell K_\ell \tag{1.13}$$

where the last equality follows from the induction hypothesis. Rearranging (1.13) yields (1.11) with $j = i + 1$ and the proof is complete. ■

We verify (1.10) indirectly. Let

$$f_\sigma(t) = e^{-\sigma t} \ . \tag{1.14}$$

If $\sigma > 0$

$$(G_\delta f_\sigma)(t) = \frac{1}{\delta}\int_t^\infty e^{(t-s)/\delta} e^{-\sigma s} ds = \frac{1}{1 + \sigma\delta} e^{-\sigma t} \tag{1.15}$$

from which it follows that

$$(A_P f_\sigma)(t) = \prod_{\ell=1}^{i} (1 + \sigma\delta_\ell)^{-1} \quad \text{for} \quad t_{i-1} \le t < t_i \tag{1.16}$$

and hence

$$\int_0^\infty (A_P f_\sigma)(s)ds = \sum_{i=1}^\infty \delta_i \prod_{\ell=1}^{i} (1+\sigma\delta_\ell)^{-1} . \tag{1.17}$$

Setting $r_i = (1+\sigma\delta_i)^{-1}$ we claim that

$$\delta_1 r_1 + \delta_2 r_2 r_1 + \cdots + \delta_i r_i r_{i-1} \cdots r_2 r_1 + \sigma^{-1} r_i r_{i-1} \cdots r_2 r_1 = \sigma^{-1} \tag{1.18}$$

for $i = 1,2,\ldots$. The proof parallels the proof of Lemma 1. Since $\sum_{i=1}^\infty \delta_i = \infty$, we have $r_i r_{i-1} \cdots r_1 \to 0$ as $i \to \infty$ and (1.17), (1.18) together imply

$$\int_0^\infty (A_P f_\sigma)(s)ds = \frac{1}{\sigma} = \int_0^\infty f_\sigma(s)ds .$$

Setting $f = f_\sigma$ in (1.8), letting $i \to \infty$ and using the above implies (1.10).

It remains to verify (6). (We remark that the previous results did not require $\mu(P) < \infty$). In view of (4), which is independent of P, it suffices to verify (6) for a dense subset F of $L^1(0,\infty)$. It is convenient to choose $F = \text{span}\{f_\sigma : \sigma > 0\}$. (In fact, $\text{span}\{e^{-nt} : n = 1,2,\ldots\}$ is dense in $L^1(0,\infty)$, as is well known. To see this, use the change of variables $x = e^{-t}$ which exchanges $(0,\infty)$ and $(0,1)$ while e^{-nt} becomes x^n.) To proceed, we estimate $|A_P f_\sigma - f_\sigma|$ in terms of $\mu(P)$. For convenience of future referencing the simple lemma which does so is stated without using the notation above.

<u>Lemma 2</u>: Let $\{\delta_i\}_{i=1}^\infty$ be a sequence of positive numbers satisfying $\sum_{i=1}^\infty \delta_i = \infty$ and $\sigma > 0$. Let $t_0 = 0$, $t_i = \delta_1 + \delta_2 + \cdots + \delta_i$ for $i = 1,2,\ldots$ and $\mu = \sup_{1 \le i < \infty} \delta_i$. If

$$g(t) = \prod_{\ell=1}^{i} (1+\sigma\delta_\ell)^{-1} \quad \text{for} \quad t_{i-1} \le t < t_i$$

then

$$|g(t) - e^{-\sigma t}| \le e^{-\sigma t}\max\{e^{\mu\sigma t}e^{\sigma^2\mu^2} - 1,\ 1 - e^{-\sigma\mu}\} .$$

<u>Proof of Lemma 2</u>: It is enough to treat $\sigma = 1$, for then the general result follows upon replacing $\{\delta_i\}$ by $\{\sigma\delta_i\}$ and t by σt. Elementary calculus yields

$$e^{-\delta} \le (1+\delta)^{-1} \le e^{-\delta+\delta^2} \quad \text{for} \quad \delta > 0 . \tag{1.19}$$

Multiplying these inequalities yields

$$(1.20)\qquad e^{-(\delta_1+\cdots+\delta_i)} \le \prod_{\ell=1}^{i} (1+\delta_\ell)^{-1} \le e^{-(\delta_1+\cdots+\delta_\ell)} e^{(\delta_1^2+\cdots+\delta_\ell^2)} .$$

Using (1.20) and the inequalities

$$\delta_1^2 + \cdots + \delta_i^2 \le \mu(\delta_1 + \cdots + \delta_i) \quad \text{and} \quad t \le \delta_1 + \cdots + \delta_i \le t + \mu$$

for $t_{i-1} \le t < t_i$ establishes

$$e^{-t}(e^{-\mu} - 1) \le g(t) - e^{-t} \le e^{-t}(e^{\mu(t+\mu)} - 1)$$

and hence Lemma 2 in the case $\sigma = 1$. ■

<u>End of proof of the Theorem</u>: By (1.16) and Lemma 2

$$\int_0^\infty |A_P f_\sigma - f_\sigma| ds \le \int_0^\infty e^{-\sigma s} \max\{e^{\mu\sigma s} e^{\sigma^2\mu^2} - 1,\ 1 - e^{-\sigma\mu}\} ds$$

where $\mu = \mu(P)$. The right hand side above tends to zero as $\mu \to 0$, and the proof is complete. ■

<u>Remark</u>: If $m \in L^1(-\infty,0) \cap L^\infty(-\infty,0)$, $m_\delta(r) = \delta^{-1} m(r/\delta)$ and (1) is replaced by $G_\delta f = m_\delta \circ f$, the first part of the above proof adapts to establish that

$$\int_0^\infty |A_P f(s)| ds \le C \int_0^\infty |f(s)| ds$$

provided $C \ge 0$ can be chosen to satisfy

$$|m(r)| + C \int_r^0 |m(s)| ds \le C \quad \text{a.e.} \quad -\infty < r \le 0 .$$

This last estimate is equivalent to $|m_\delta| + |m_\delta| * C \le C$. If $||m||_{L^1(-\infty,0)} < 1$ we may set

$$C = ||m||_{L^\infty(-\infty,0)} / (1 - ||m||_{L^1(-\infty,0)}) .$$

(Consideration of the case $m(r) = 2e^r$ shows some such restriction is necessary.) By contrast, our proofs of (5) and (6) required special properties of the exponential kernel.

Section 2. Motivation

Let X be a Banach space and A be an accretive operator in X (see, e.g., [1], [2] for terminology). If $f \in L^1(0,\infty;X)$ and $x \in X$, we say that

$$\text{(2.1)} \qquad \begin{cases} u' + Au \ni f \\ u(0) = x \end{cases}$$

has an ε-approximate solution on $[0,T]$ if there are finite sequences $0 = t_0 < t_1 < \cdots < t_N$, and $\{f_1,f_2,\ldots,f_N\}$, $\{x_0,x_1,\ldots,x_N\} \subset X$ such that

$$\text{(2.2)} \qquad \frac{x_{i+1} - x_i}{t_{i+1} - t_i} + Ax_{i+1} \ni f_{i+1}, \qquad i = 0,1,\ldots,N$$

and

$$\text{(2.3)} \qquad t_N \geq T,\ t_{i+1} - t_i < \varepsilon,\ \|x_0 - x\| < \varepsilon \text{ and } \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|f_{i+1} - f(s)\| ds < \varepsilon .$$

In this case, the step function whose value on $(t_i,t_{i+1}]$ is x_{i+1} is called an ε-approximate solution of (2.1). It is shown in [3] that if (2.1) has an ε-approximate solution on $[0,T]$ for each $\varepsilon > 0$, then these solutions converge uniformly as $\varepsilon \downarrow 0$ to a unique limit $u \in C([0,T];X)$ which is the solution (in a certain sense) of (2.1). The estimates which prove this are considerably more involved in the case that $f \not\equiv 0$ than in the simpler case $f \equiv 0$. The approximation theorem proved in this note allows us to reduce the problem (2.1) with a general $f \in L^1(0,\infty;X)$ to the case $f \equiv 0$ in the following way: Define an operator $\mathcal{A}$ in $X \times L^1(0,\infty;X)$ by

$$D(\mathcal{A}) = D(A) \times W^{1,1}(0,\infty;X)$$

and

$$\mathcal{A}(x,g) = \{(y - g(0),-g') : y \in Ax\} .$$

We show below that $\mathcal{A}$ is accretive. Given an ε-approximate solution of (2.1) on $[0,T]$ as above, define $\{g_0,g_1,\ldots,g_N\} \subset W^{1,1}(0,\infty;X)$ by $g_0 = f$, $\delta_i = t_i - t_{i-1}$, $g_{i+1} = G_{\delta_{i+1}} g_i$. Then the function whose value on $(t_i,t_{i+1}]$ is (x_{i+1},g_{i+1}) is an

$\varepsilon + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|g_{i+1}(0) - f(s)\| ds$ approximate solution of

$$(2.4)\qquad \begin{cases} U' + AU \ni 0 \\ U(0) = (x,f)\ . \end{cases}$$

By the (clearly valid) version of Theorem 1 in which $L^1(0,\infty)$ is replaced by $L^1(0,\infty;X)$ the term

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|g_{i+1}(0) - f(s)\|ds$$

tends to zero as $\mu = \max_i \delta_i \to 0$.

We now briefly sketch the proof that A is accretive. If Z is any Banach space, set

$$[p,q]_Z = \lim_{\lambda \downarrow 0} \frac{\|p + \lambda q\|_Z - \|p\|_Z}{\lambda} = \inf_{\lambda>0} \frac{\|p + \lambda q\|_Z - \|p\|_Z}{\lambda}\ .$$

If $Y = X \times L^1(0,\infty;X)$ is equipped with the norm

$$\|(x,g)\|_Y = \|x\|_X + \int_0^\infty \|g(s)\|_X ds \quad \text{for} \quad (x,g) \in X$$

then one computes that

$$[(x,g),(y,h)]_Y = [x,y]_X + \int_0^\infty [g(s),h(s)]_X ds\ .$$

It follows that A is accretive in Y if for every $(x,y),(\hat{x},\hat{y}) \in A$ and $g,\hat{g} \in W^{1,1}(0,\infty;X)$ we have

$$(2.5)\quad [(x-\hat{x},g-\hat{g}),((y-g(0)) - (\hat{y}-\hat{g}(0)),-g'+\hat{g}')]_Y$$

$$= [x-\hat{x},(y-\hat{y}) - (g(0)-\hat{g}(0))]_X + \int_0^\infty [g(s)-\hat{g}(s),-g'(s)+\hat{g}'(s)]_X ds \ge 0\ .$$

The first term in (2.5) is estimated by

$$(2.6)\quad [x-\hat{x},(y-\hat{y}) - (g(0)-\hat{g}(0))]_X \ge [x-\hat{x},(y-\hat{y})]_X - \|g(0)-\hat{g}(0)\|_X \ge - \|g(0) - \hat{g}(0)\|_X\ ,$$

where the first inequality is due to the fact that $[p,q]_Z$ is Lipschitz in q with constant 1 and the second inequality is because $(x,y),(\hat{x},\hat{y}) \in A$ and A is accretive. The second term in (2.5) is given by

$$(2.7)\quad \int_0^\infty [g(s)-\hat{g}(s),\ -g'(s)+\hat{g}'(s)]_X ds = \int_0^\infty -\frac{d}{ds}\|g(s)-\hat{g}(s)\|_X ds = \|g(0) - \hat{g}(0)\|_X$$

since $\frac{d}{ds} \|k(s)\|_X = [k(s), k'(s)]_X$ wherever $\|k(s)\|_X$ and $k(s)$ are both differentiable. Together, (2.6) and (2.7) imply (2.5) and hence that A is accretive.

The system (2.4) was introduced in [4] for another purpose. The results we have just obtained reduce many questions concerning (2.1) to the same questions for $f = 0$.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 1911	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) AN APPROXIMATION OF INTEGRABLE FUNCTIONS BY STEP FUNCTIONS WITH AN APPLICATION.		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) M. G. Crandall ~~and~~ A. Pazy		8. CONTRACT OR GRANT NUMBER(s) (15) DAAG29-75-C-0024, NSF-MCS78-01245
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit # 6 - Spline Functions and Approximation Theory
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below. (12) 13p.		12. REPORT DATE (11) January 1979
		13. NUMBER OF PAGES 8
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (14) MRC-TSR-1911		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. (9) Technical summary rept.,

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Step functions
Approximation theory
Accretive operators

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $f \in L^1(0,\infty)$, $\delta > 0$ and $(G_\delta f)(t) = \delta^{-1} \int_t^\infty e^{(t-s)/\delta} f(s)ds$. Given a partition $P = \{0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots\}$ of $[0,\infty)$ where $t_i \to \infty$, we approximate f by the step function $A_P f$ defined by

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20. ABSTRACT - Cont'd.

$$A_P f(t) = (G_{\delta_i} G_{\delta_{i-1}} \cdots G_{\delta_1} f)(0) \quad \text{for} \quad t_{i-1} \leq t < t_i ,$$

where $\delta_i = t_i - t_{i-1}$. The main results concern several properties of this process, with the most important one being that $A_P f \to f$ in $L^1(0,\infty)$ as $\mu(P) = \sup_i \delta_i \to 0$. An application to difference approximations of evolution problems is sketched.